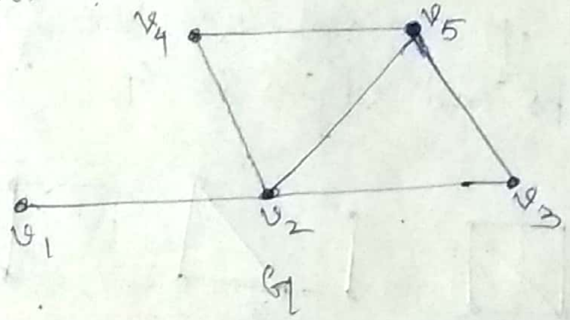
Connected Graphs.

A **walk** of a graph G is an alternating sequence of points and edges $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ beginning and ending with points (vertex), in which ~~edge~~ each edge is incident with the two vertex immediately preceeding and following it. This walk joins v_0 and v_n and sometimes called $v_0 - v_n$ walk.

It is **closed** if $v_0 = v_n$ and is **open** otherwise.

It is a **trail** if all the lines distinct and a **path** if all the points (and thus necessarily all the lines) are distinct.

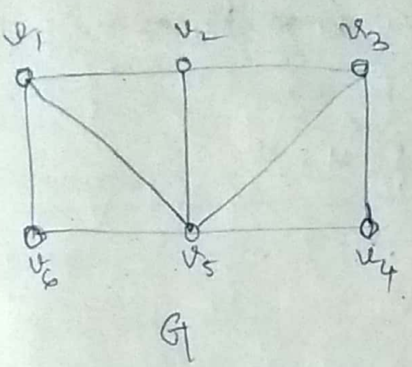
If the walk is closed then it is a **cycle** provided its n points are distinct and $n \geq 3$.

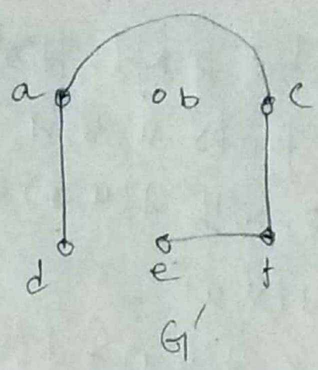


In the graph G, $v_1 v_2 v_5 v_2 v_3$ is a walk which is not a trail and $v_1 v_2 v_5 v_4 v_2 v_3$ is a trail which is not a path; $v_1 v_2 v_5 v_4$ is a path and $v_2 v_4 v_5 v_2$ is a cycle.

We denote by C_n the graph consisting of a cycle with n points and by P_n a path with n points. C_3 is called a **triangle**.

A graph is **connected** if every pair of points are joined by a path.





For example, G is a connected graph and G' is disconnected.

The Length of a walk $v_0, v_1 \cdots v_n$ is n, the number of occurence of lines in it.

The girth of a graph G, denoted $g(G)$, is the length of a shortest cycle (if any) in G.

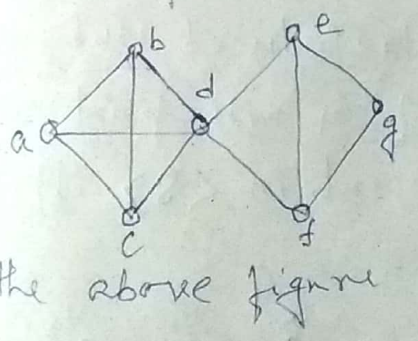
The circumference $c(G)$ is the length of any longest cycle.

Note that these terms are undefined if G has no cycles.

The distance $d(u, v)$ between two points u and v in G is the length of a shortest path joining them (if any).

otherwise $d(u,v) = \infty$.

For example



in the above figure $d(a,f) = 2$ and $d(a,d) = 1$

~~In a connec~~

In a connected graph, distance is a metric.

i.e, for all points u, v, w

1) $d(u,v) \geq 0$, with $d(u,v) = 0$ iff $u = v$

2) $d(u,v) = d(v,u)$

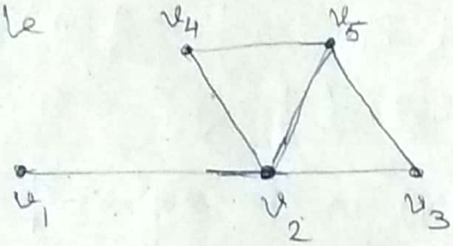
3) $d(u,w) \leq d(u,v) + d(v,w)$

A shortest u-v path is often called a geodesic.

The diameter $d(G)$ of a connected graph G is the length of any longest geodesic.

For example

G:



$(v_2 v_4 v_5 v_2)$

The above graph G, girth $g(G) = 3$, circumference $c(G) = 4$ $(v_2 v_4 v_5 v_3 v_2)$ and diameter $d(G) = 2$.

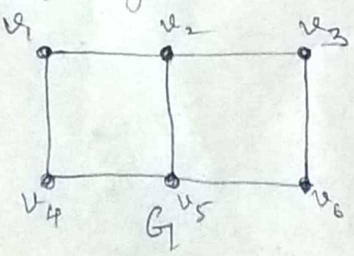
An invariant of a graph G is a number associated with G which has the same value for any graph isomorphic to G.

Subgraph (cont'd)

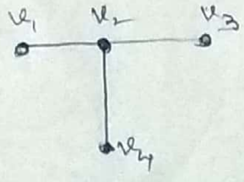
If G_1 is a subgraph of G, then G is a supergraph of G_1.

A spanning subgraph is a subgraph containing all the points of G.

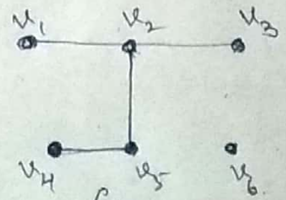
For any set S of points of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with point set S. Thus two points of S are adjacent in $\langle S \rangle$ iff they are adjacent in G.



G



G_1



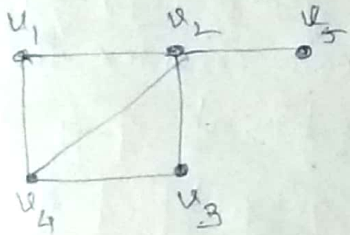
G_2

G_2 is a spanning subgraph of G but G_1 is not.
G_1 is an induced subgraph but G_2 is not.

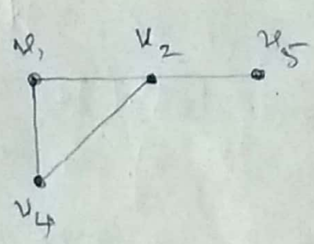
The removal of a point v_i from a graph G results in that subgraph $G - v_i$ of G consisting of all points of G except v_i and all edges not incident with v_i.
Thus $G - v_i$ is the maximal subgraph of G not containing v_i.

On the other hand removal of a edge e_j from G yields the spanning subgraph $G - e_j$ containing all edges of G except e_j. Thus $G - e_j$ is the maximal subgraph of G not containing e_j.

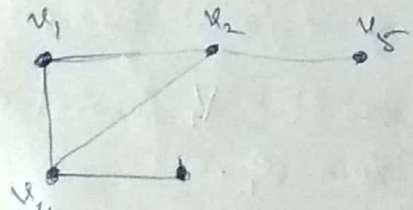
G:

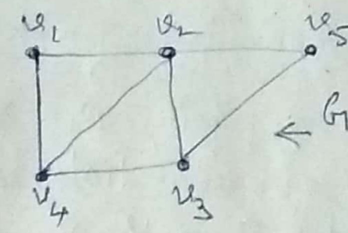


$G - v_3$:



$G - v_2v_3$:





← $G + v_3v_5$

If v_i & v_j are not adjacent in G, the addition of line v_iv_j results in the smallest subgraph of G containing the line v_iv_j.

The following observation can be made for subgraph.

i) Every graph is its own subgraph.
ii) A subgraph of a subgraph of G is a subgraph of G.
iii) A single vertex in a graph G is a subgraph of G.
iv) A single edge in G, together with its end vertices is also a subgraph of G.

Th. A graph is bipartite if and only if all its cycles are even.

Proof.

If G is a bigraph, then its point set V can be partitioned into two sets V_1 and V_2 so that every line of G joints a point V_1 with a point of V_2. Thus all the odd subscripted vertices in every cycle $v_1 v_2 \cdots v_n v_1$ in G belongs to the set V_1 (say). and the others in V_2. So that its length n is even.

Next let us assume that G be a graph with all its cycles are even. Without loss of generality, let G be connected. (for otherwise we can consider the components of G seperately) [component of G is defined as a maximal connected subgraph of G. Thus a disconnected graph has at least two components.]

Take any point $v_1 \in V$, and let V_1 consist of v_1 ~~all~~ and all points at even distance from v_1.

Also take $V_2 = V - V_1$. Since all the cycles of G are even, every lines of G joins a point of V_1 with a point of V_2. ~~and~~ For suppose there is a line uv joining two points of V_1. Then the union of geodesics from v_1 to v and from v_1 to u together with the line uv contains an odd cycle, a contradiction.

Hence the graph is bipartite.

Hence proved.